Efficient Subjective Bayesian Network Belief Propagation for Trees

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Motivation

- Decision making requires understanding of uncertainty
- Probabilistic reasoning characterizes knowledge of unobserved variables based upon past history
- Many application entail probabilistic reasoning with limited training
- Bayesian network with conditional probabilities provided by
  - Limited statistical evidence
  - Subject matter experts with limited experience in the domain
- Cannot depend on "Big Data"
- Past efforts:
  - Valuation Based Systems (based on Evidence Theory)
  - Subjective Bayesian Network: Second-order Bayesian reasoning demonstrated over three node networks

Subjective Logic

Belief Representation

Multinomial Opinions

$$\omega_x = \{\mathbf{b}, u, \mathbf{a}, W\}$$

$$u + \sum_{k=1}^{K} b_k = 1, \quad \sum_{k=1}^{K} b_k = 1$$

Belief	Uncer.	Base Rate
$b_k \geq 0$	$u \geq 0$	$a_k \geq 0$

Noninformative Prior Weight : W

$$\alpha_k = \frac{W}{u} b_k + W a_k$$

1-to-1

$$b_k = \frac{1}{S}(\alpha_k - W a_k)$$

$$u = \frac{W}{S}$$

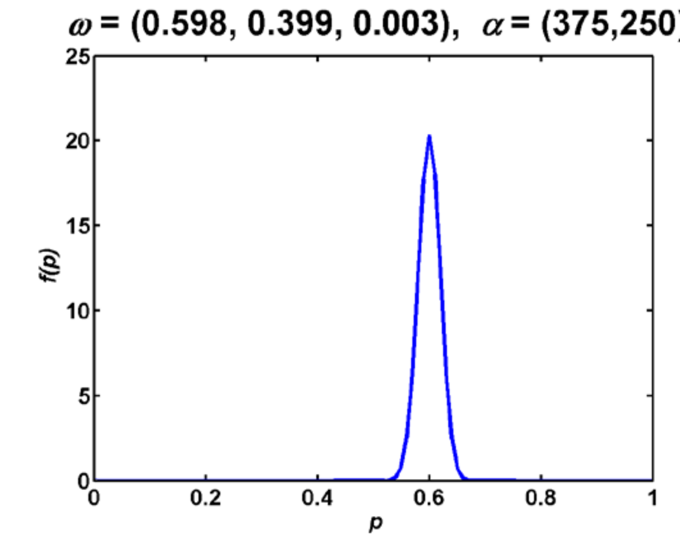
Statistical Representation

Dirichlet Distribution

$$f_\beta(\mathbf{p} \mid \boldsymbol{\alpha})$$

Strength : $S = \sum_{k=1}^{K} \alpha_k$

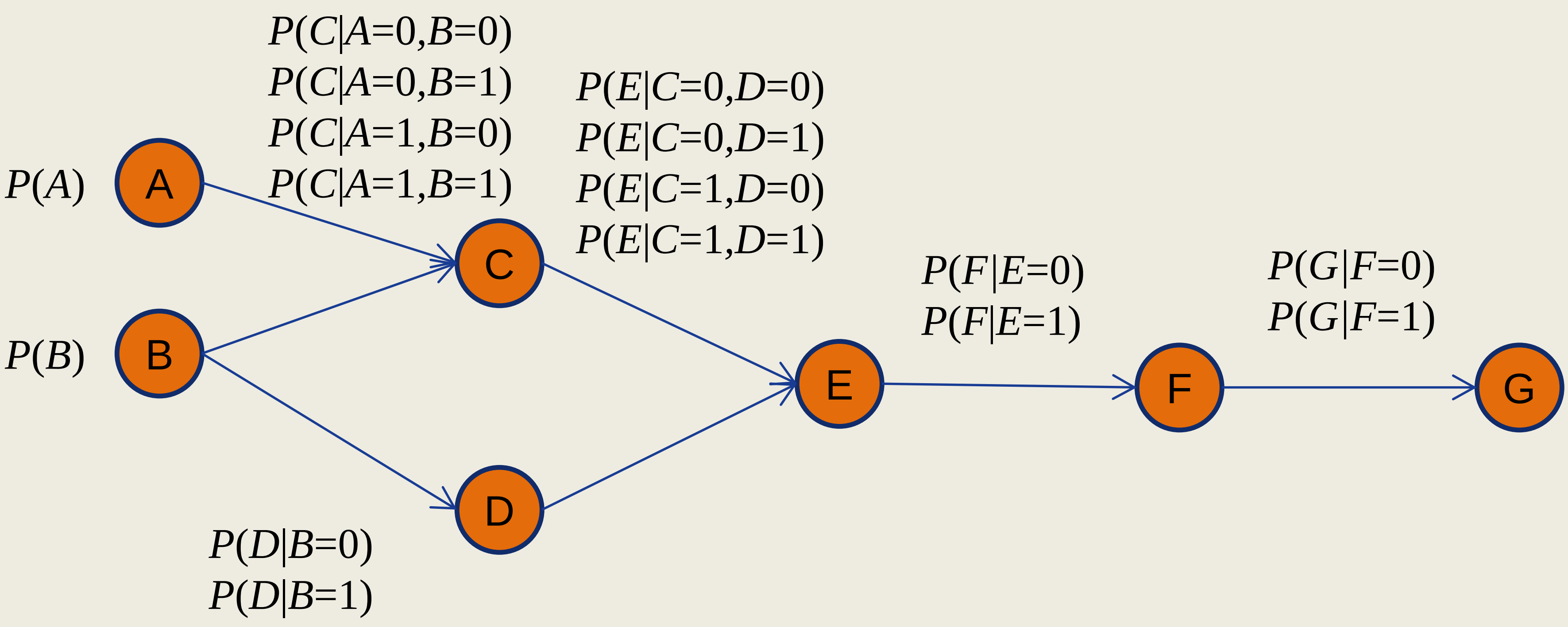
Evidence : $\boldsymbol{\alpha} - W\mathbf{a}$



Expected (or Projected) probability

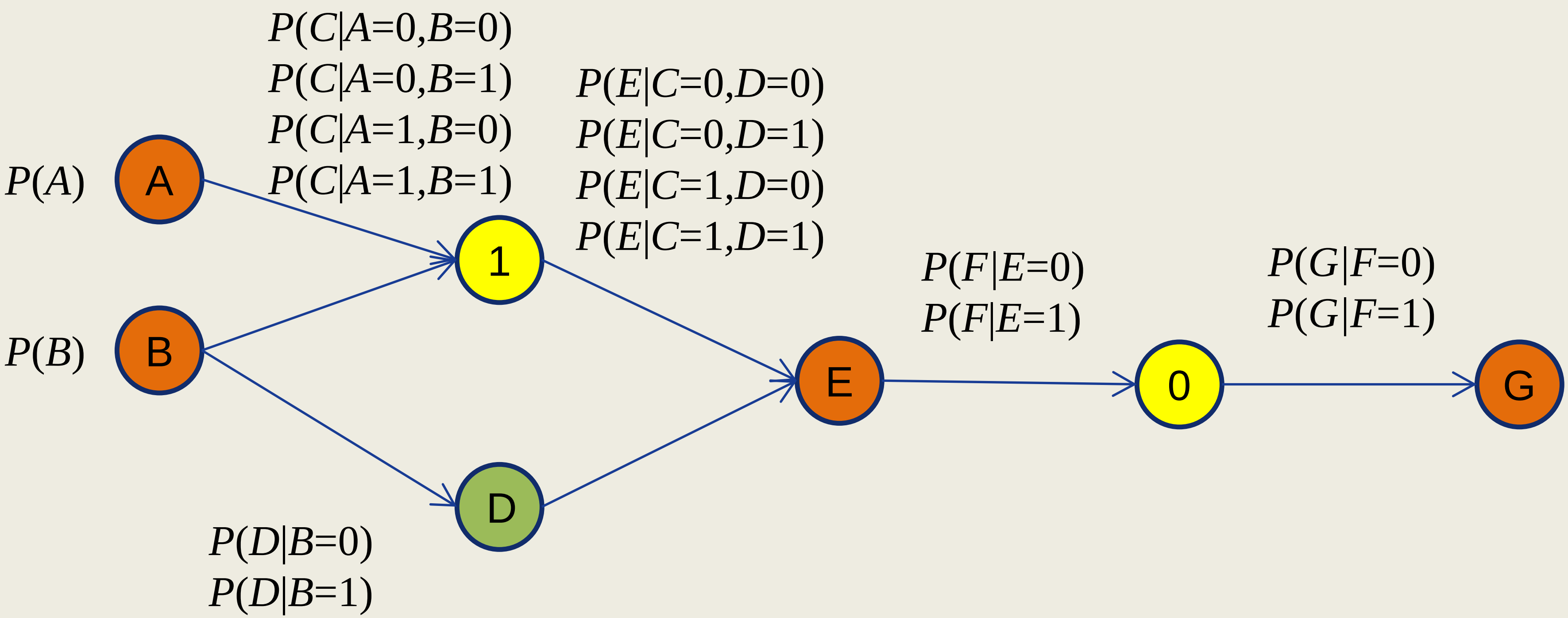
$$m_k = E[p_k] = b_k + a_k u = \frac{\alpha_k}{S}$$

Bayesian Network



$$P(A,B,C,D,E,F,G) = P(G|F)P(F|E)P(E|C,D)P(D|B)P(C|A,B)P(B)P(A)$$

Bayesian Network Inference



$$P(D \mid C=1, F=0) = \frac{\sum_{A,B,E} P(F=0,E)P(E \mid C=1,D)P(D \mid B)P(C=1 \mid A,B)P(B)P(A)}{\sum_{A,B,D,E} P(F=0,E)P(E \mid C=1,D)P(D \mid B)P(C=1 \mid A,B)P(B)P(A)}$$

Sum-Product Algorithm
Variable Elimination

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Subjective Network

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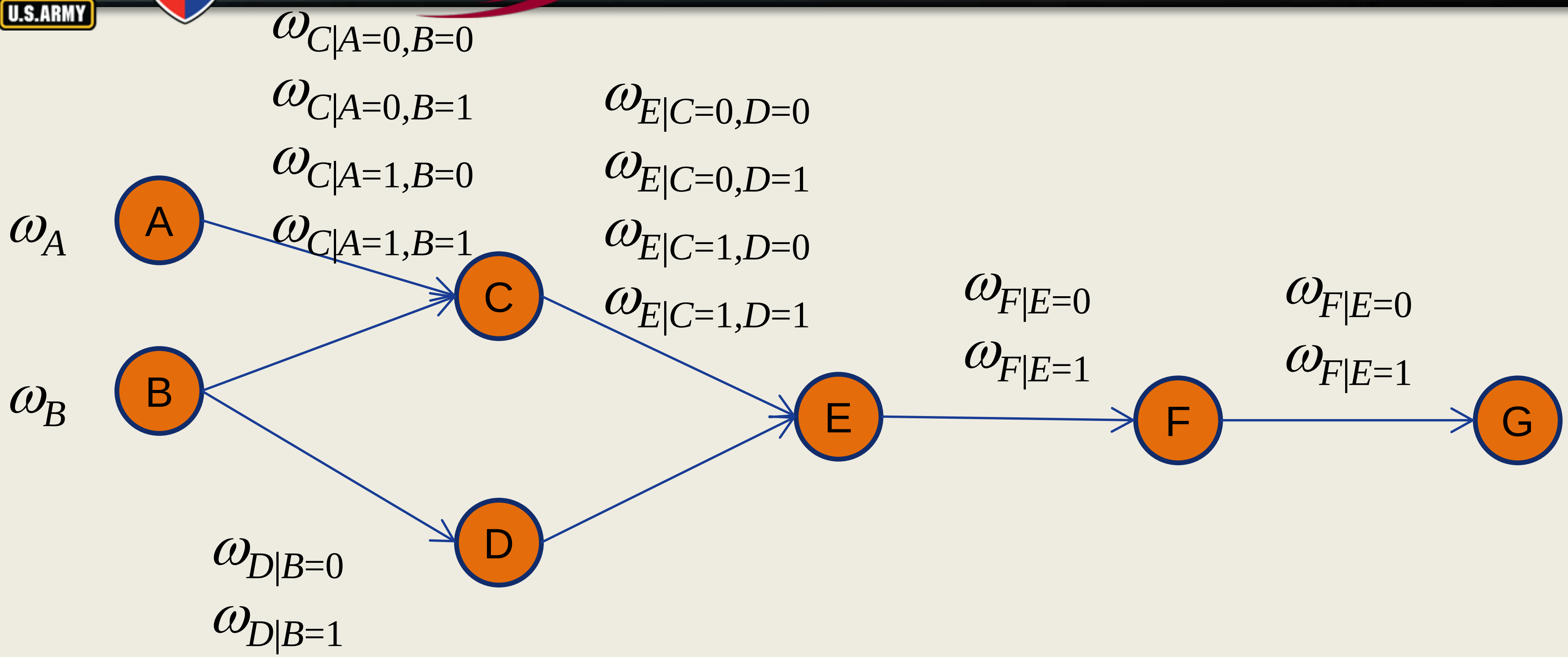
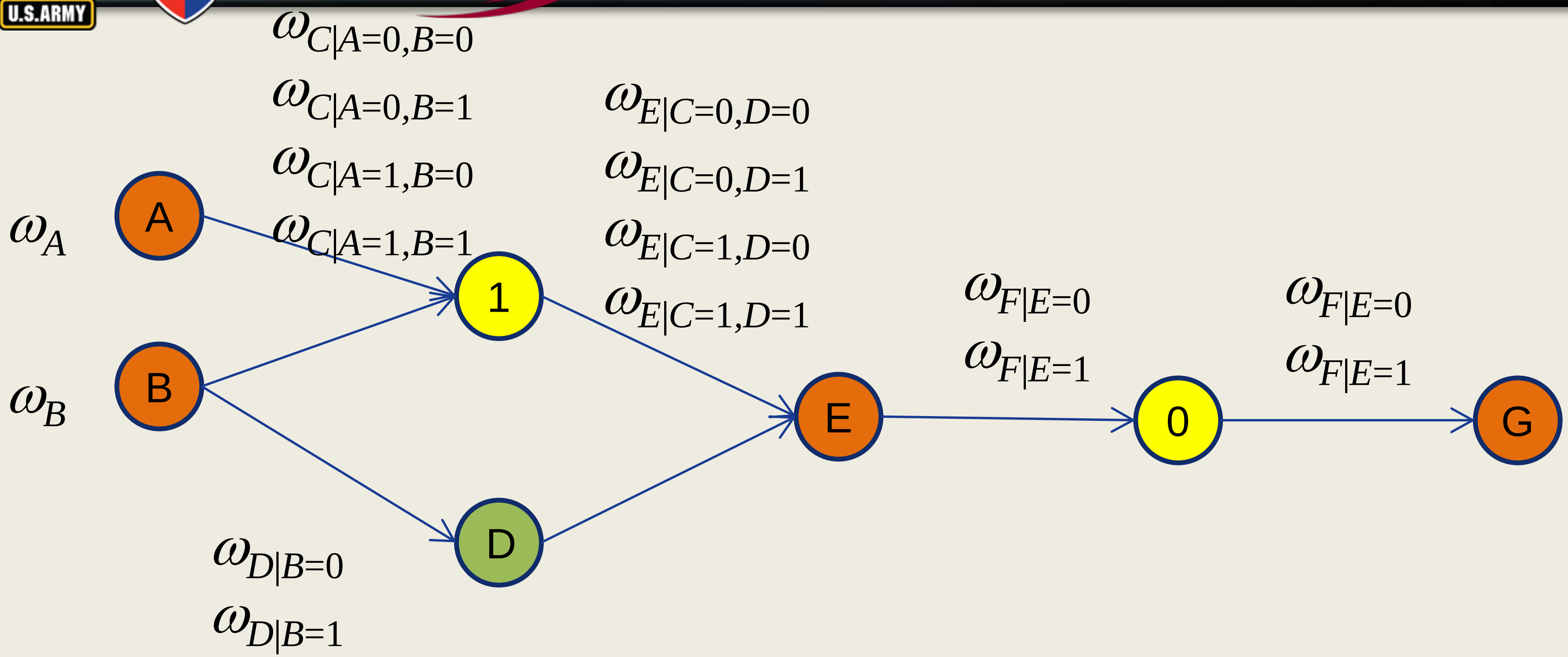


$$P(A,B,C,D,E,F,G) = P(G|F)P(F|E)P(E|C,D)P(D|B)P(C|A,B)P(B)P(A)$$

Random Variables

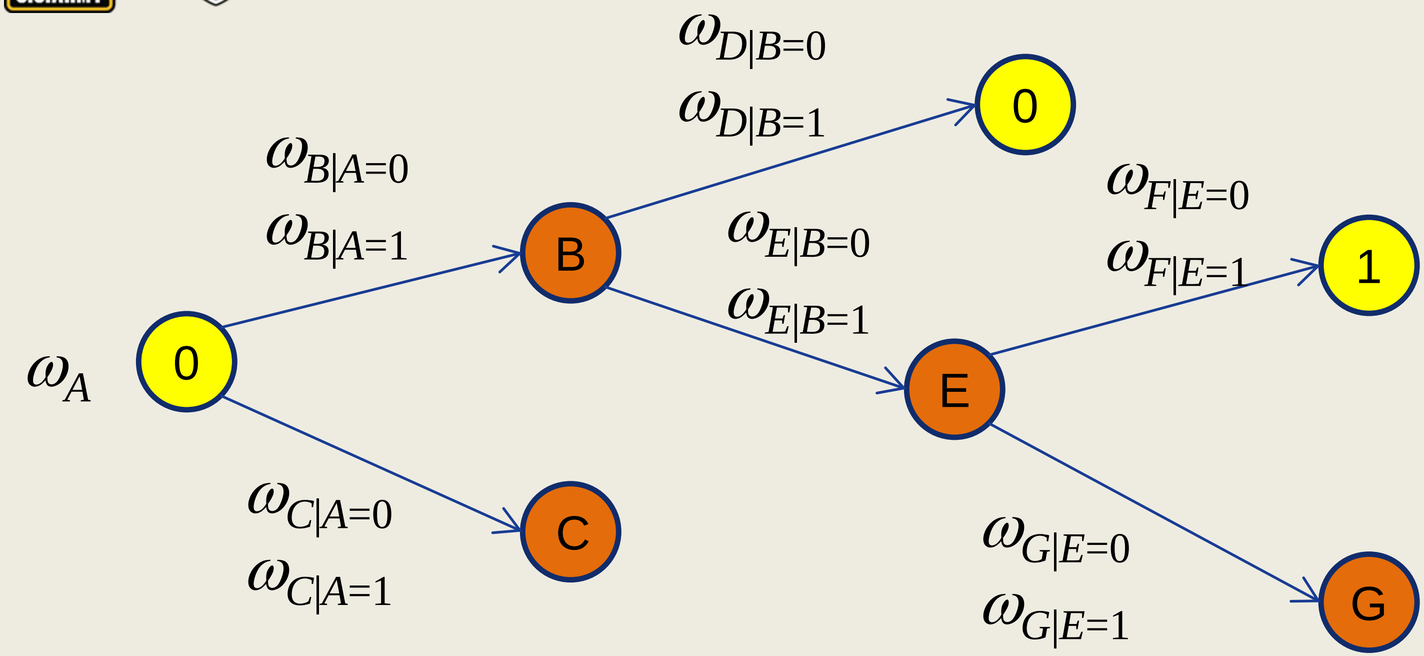
Subjective Network Inference



$$P(D \mid C=1, F=0) = \frac{\sum_{A,B,E} P(F=0,E)P(E \mid C=1,D)P(D \mid B)P(C=1 \mid A,B)P(B)P(A)}{\sum_{A,B,D,E} P(F=0,E)P(E \mid C=1,D)P(D \mid B)P(C=1 \mid A,B)P(B)P(A)}$$

Inference Algorithms (TBD) → $\omega_{D|C=1,F=0}$

Subjective Belief Propagation



1. **Approximate the maginals as beta random variables**
2. **Messages passed between nodes are approximated as beta random variables**
3. **Use moment matching to determine approximate beta distributions**
4. **When needed, use first order Taylor series approximation to compute the moments**

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Belief Propagation

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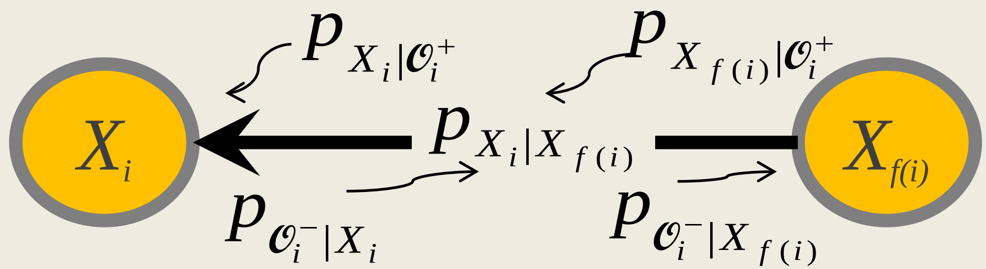
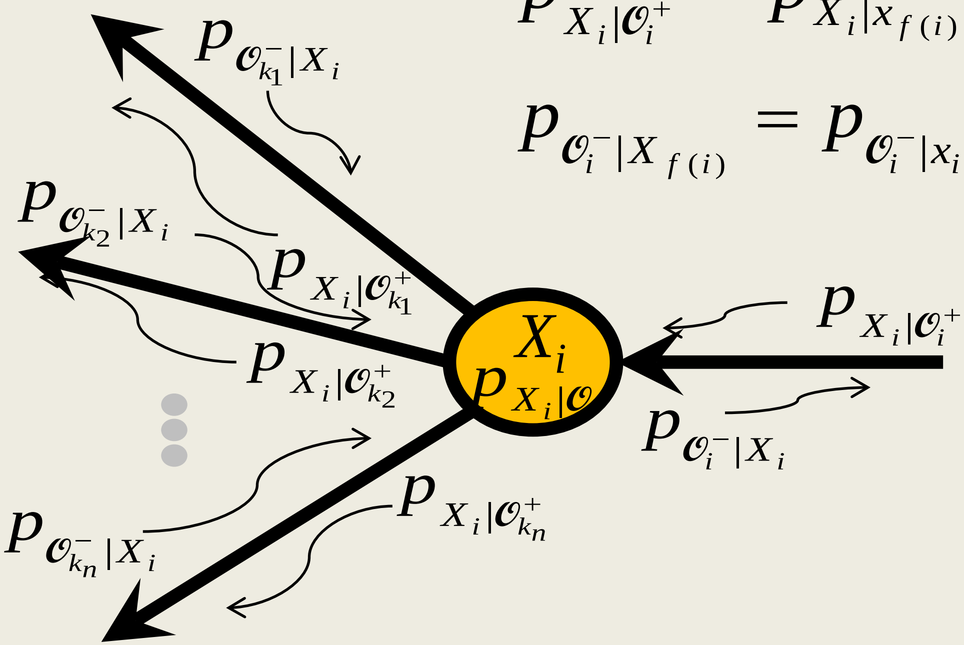


$$p_{X_i|\mathcal{O}_i^+} = p_{X_i|x_{f(i)}}\, p_{x_{f(i)}|\mathcal{O}_i^+} + p_{X_i|\bar{x}_{f(i)}} \left(1 - p_{x_{f(i)}|\mathcal{O}_i^+}\right)$$

$$p_{\mathcal{O}_i^-|X_{f(i)}} = p_{\mathcal{O}_i^-|x_i}\, p_{x_i|X_{f(i)}} + p_{\mathcal{O}_i^-|\bar{x}_i} \left(1 - p_{x_i|X_{f(i)}}\right)$$



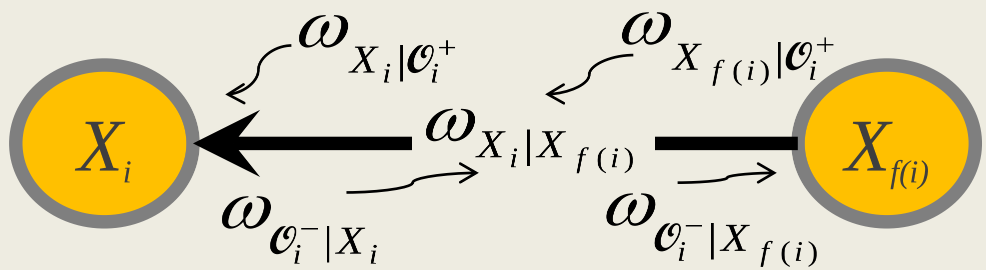
$$p_{X_i|\mathcal{O}_{k_1}^+} = \frac{1}{C}\, p_{X_i|\mathcal{O}_i^+} \prod_{j=2}^{n} p_{\mathcal{O}_{k_j}^-|X_i}$$

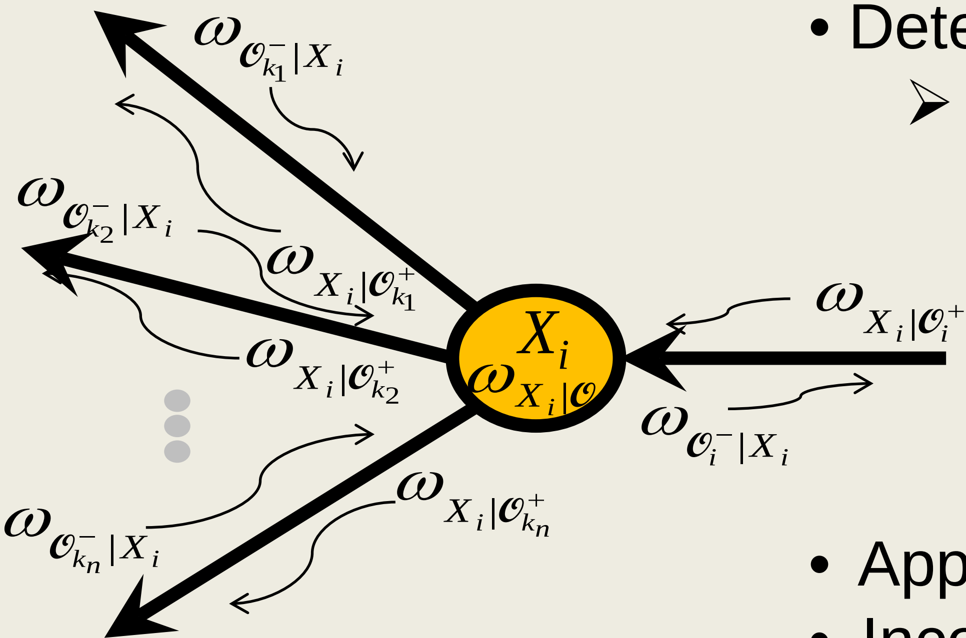
$$C = \sum_{X_i \in \{x_i, \bar{x}_i\}} p_{X_i|\mathcal{O}_i^+} \prod_{j=2}^{n} p_{\mathcal{O}_{k_j}^-|X_i}$$

Subjective Belief Propagation

Three basic operations:

1) Fusion
2) Forward propagation
3) Backward propagation

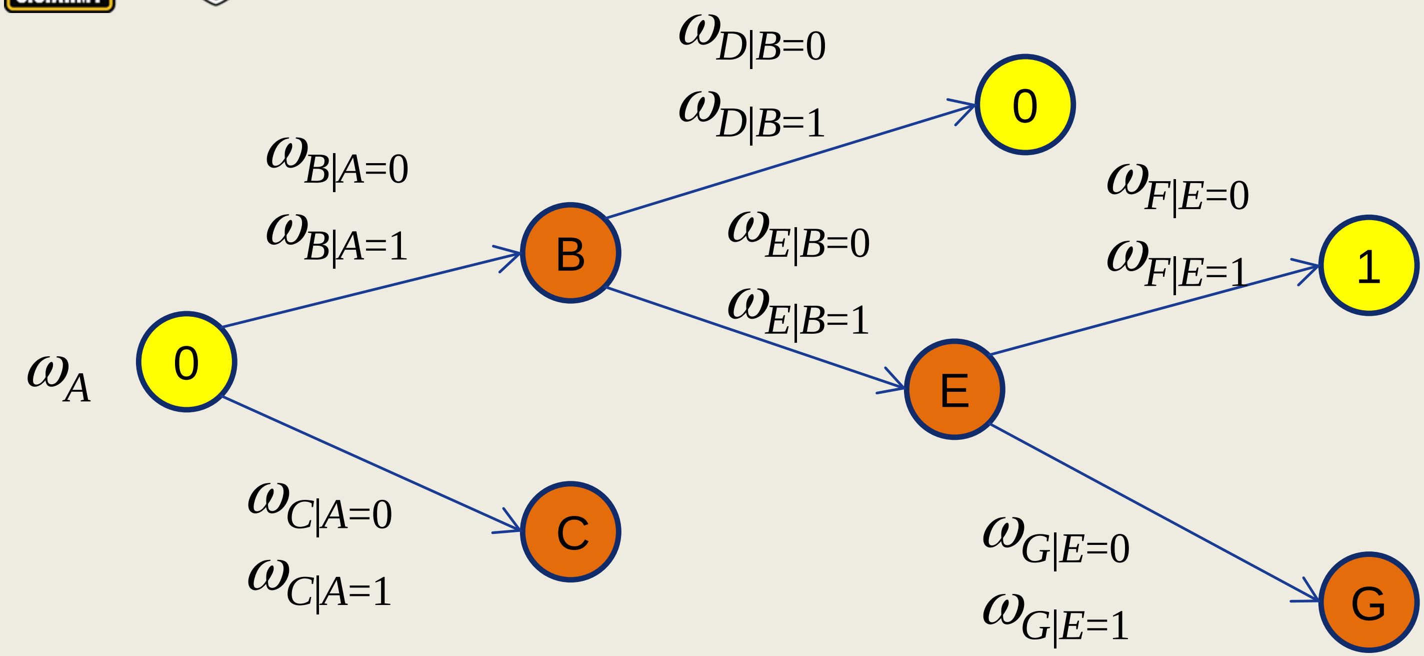




- Determine message opinions:
  - ➢ Approximate distribution of messages as beta

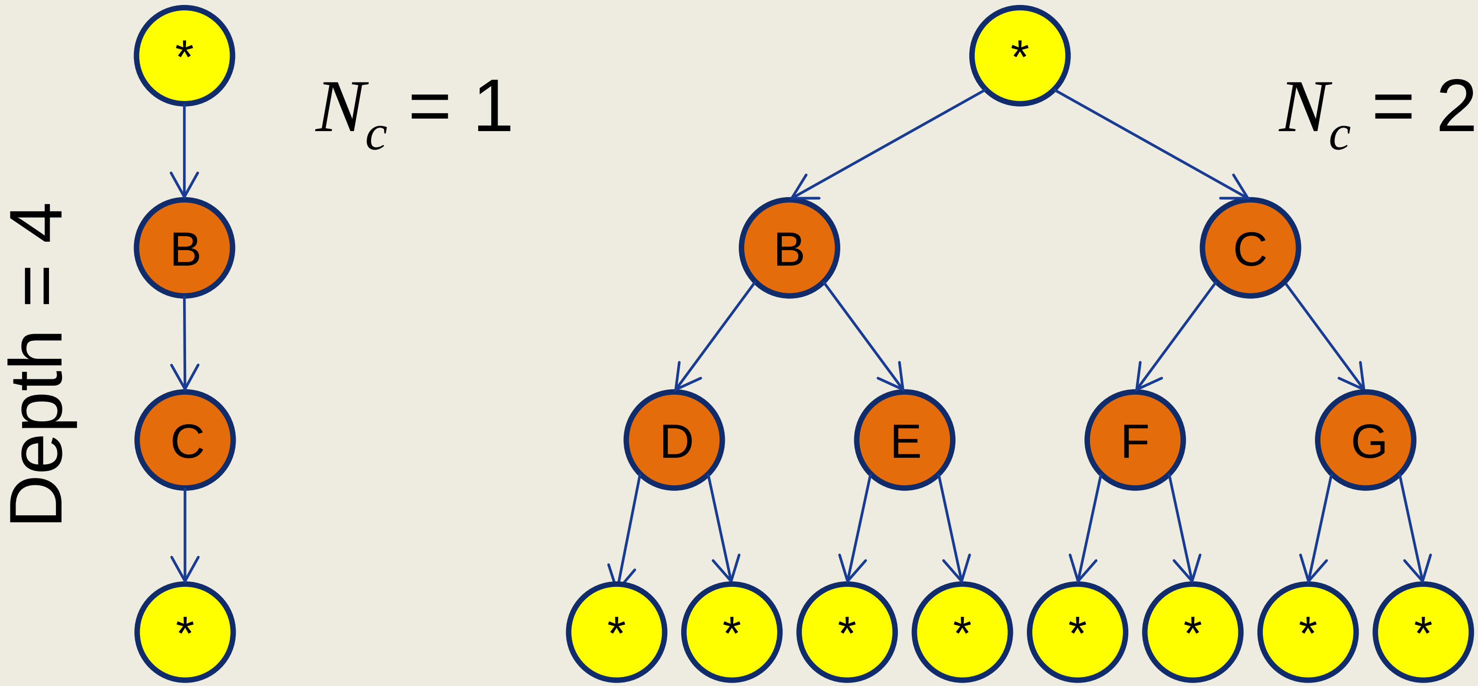
- Apply method of moments
- Incorporate first order Taylor series about mean values

Inference



1. **Backward propagation from the leaves to the root**
2. **Forward propagation from the root to the leaves**

Simulations



Various network tree structures: N_c = 1, 2, 3, 4 or 5

Various numbers of training instances: N_t = 10, 50, or 100

100 ground truth Bayesian Networks

10 trained Subjective Bayesian Networks per ground truth network

BP on Bayesian network determines ground truth marginals

SBP over SBN determines opinions for the marginals

SBP Accuracy

RMSE between mean and ground truth probabilities

N_{ins}	$N_c = 1$	$N_c = 2$	$N_c = 3$	$N_c = 4$	$N_c = 5$
10	0.178	0.202	0.236	0.256	0.269
50	0.114	0.125	0.144	0.157	0.171
100	0.084	0.088	0.105	0.119	0.132

Predicted RMSE for SBP using the derived uncertainty

N_{ins}	$N_c = 1$	$N_c = 2$	$N_c = 3$	$N_c = 4$	$N_c = 5$
10	0.203	0.219	0.227	0.225	0.216
50	0.118	0.138	0.152	0.163	0.168
100	0.088	0.093	0.110	0.128	0.137

$$VAR \approx \frac{m(1-m)u}{2}$$

VBS Accuracy

RMSE between mean and ground truth probabilities

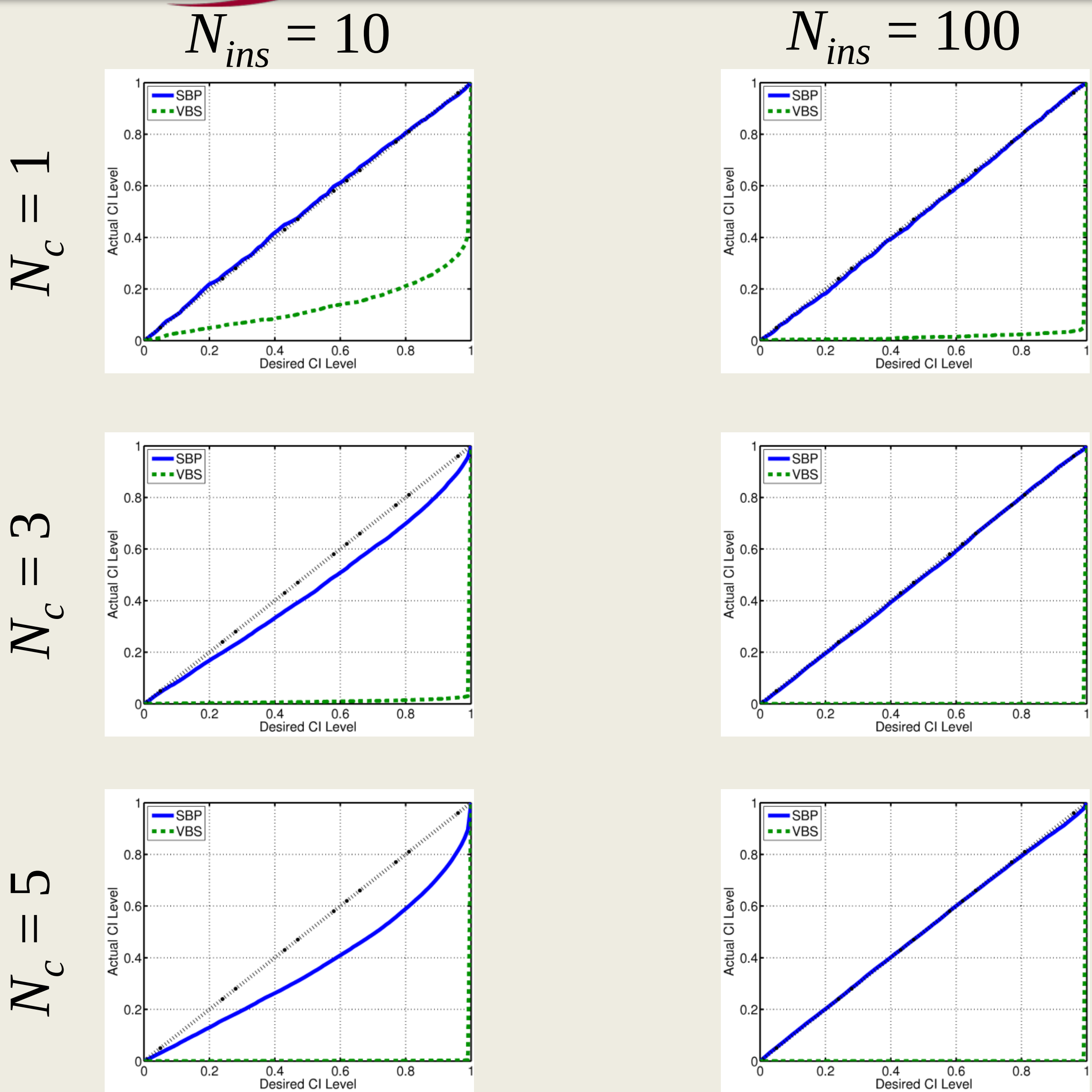
N_{ins}	$N_c = 1$	$N_c = 2$	$N_c = 3$	$N_c = 4$	$N_c = 5$
10	0.178	0.203	0.236	0.256	0.269
50	0.114	0.125	0.144	0.157	0.171
100	0.084	0.088	0.105	0.119	0.132

Predicted RMSE for VBS using the derived uncertainty

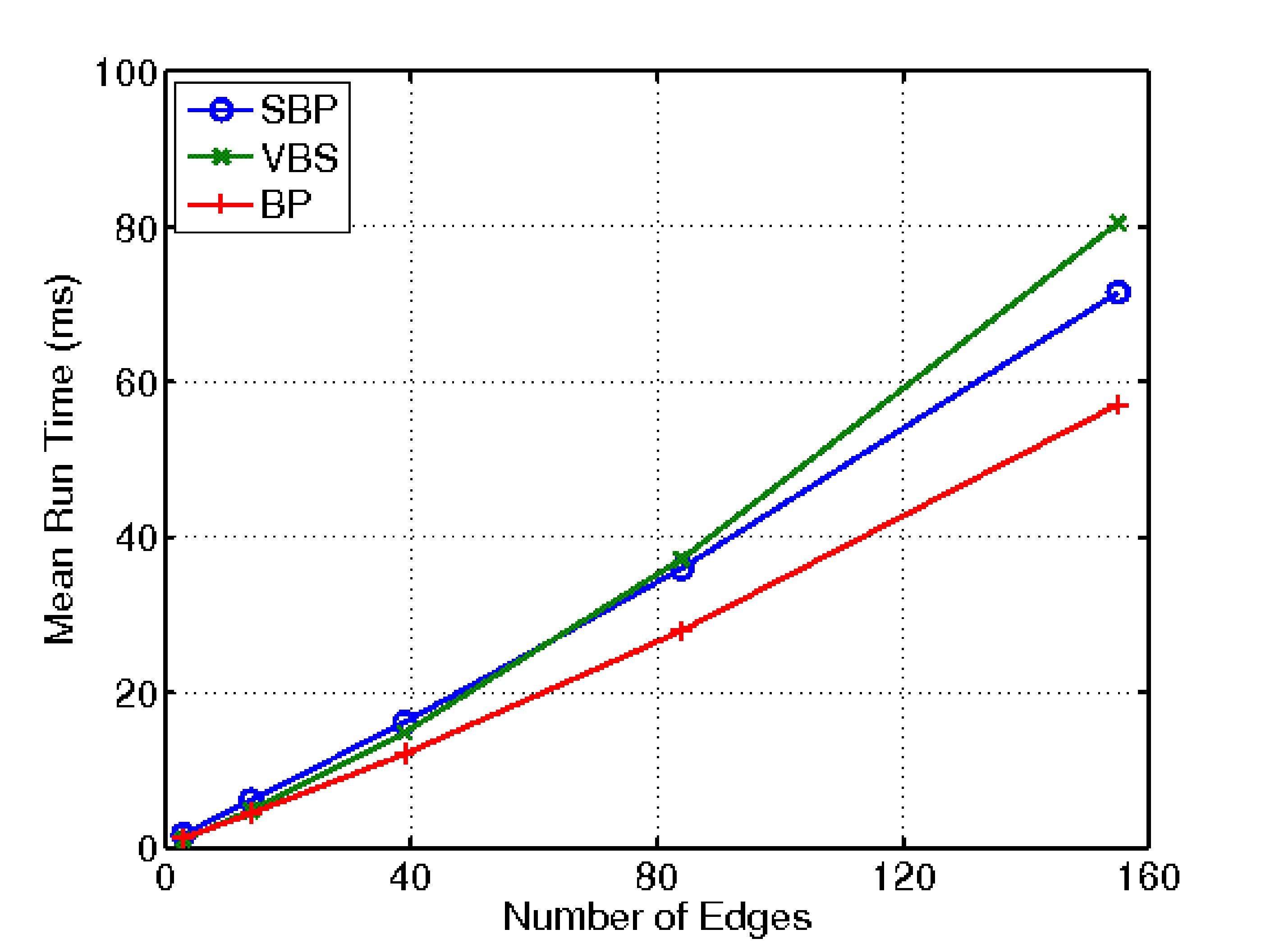
N_{ins}	$N_c = 1$	$N_c = 2$	$N_c = 3$	$N_c = 4$	$N_c = 5$
10	0.043	0.015	0.006	0.002	0.001
50	0.007	0.001	0.000	0.000	0.000
100	0.007	0.001	0.000	0.000	0.000

$$VAR \approx \frac{m(1-m)u}{2}$$

Uncertainty Accuracy



Computational Complexity



Summary

- Demonstrate inference of subjective Bayesian network
- Characterization of the uncertainty is accurate when N_c is small and when N_t sufficiently large
- Need to understand the impact of approximations:
  - Moment matching
  - Linearization to efficiently compute moments
  - Treating messages at beta random variables
- Inference over non-binary variables
- Inference over DAGs
- Joint inference and training